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A Pringsheim-type convergence criterion for continued fractions in Banach algebras

Hendrik Baumann

Department of Applied Stochastics and Operations Research, Clausthal University of Technology, Erzstr. 1, D-38678 Clausthal-Zellerfeld, Germany

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Abstract

We consider continued fractions in Banach algebras, that is

$$b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1},$$

where $(b_n)_{n\in\mathbb{N}_0}$ and $(a_n)_{n\in\mathbb{N}}$ are sequences of elements of some Banach algebra. We prove that $\left\|b_n^{-1}\right\| + \left\|a_n b_n^{-1}\right\| \leq 1$ for $n = 1, 2, \ldots$ is a sufficient condition for convergence. This result is an exact generalization of the Śleszyński–Pringsheim convergence criterion for complex continued fractions, and improves on all known results.

Keywords: Continued fractions in Banach algebras; Matrix continued fractions; Pringsheim-type convergence criterion

1. Introduction and main result

For complex continued fractions, that is

$$K = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}},$$

E-mail address: hendrik.baumann@tu-clausthal.de.

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where $b_0, b_1, \ldots, a_1, a_2 \in \mathbb{C}$, one of the most important convergence theorems is Pringsheim's criterion (see [18], chapter II, Section 14 for the proof), sometimes referred to as the Śleszyński–Pringsheim criterion. It states that

$$|b_n| \geq |a_n| + 1, \quad n = 1, 2, \ldots, \tag{$P1$}$$

is sufficient for convergence of K.

There are a lot of generalizations of continued fractions in the literature; here we consider (infinite) continued fractions of the form

$$K = b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1},$$

where $(b_n)_{n\in\mathbb{N}_0}$, $(a_n)_{n\in\mathbb{N}}$ are sequences of elements of some Banach algebra $\mathcal{R}$ with identity I. The formal definition is as follows:

Definition 1.1. Let $(b_n)_{n\in\mathbb{N}_0}$ and $(a_n)_{n\in\mathbb{N}}$ be sequences of elements of $\mathcal{R}$. Define

$$K_N^{(N)} = b_N, \quad N = 0, 1, 2, \ldots,$$
$$K_n^{(N)} = b_n + a_{n+1}\left(K_{n+1}^{(N)}\right)^{-1}, \quad n, N = 0, 1, 2, \ldots, n < N.$$

If $K_0^{(N)}$ exists for almost all $N \in \mathbb{N}$ and if $K = \lim_{N\to\infty} K_0^{(N)}$ exists, K is said to be a *convergent continued fraction*.

Naturally, we are interested in sufficient conditions for the convergence of continued fractions in Banach algebras. The main result of this paper may be considered as an exact generalization of Pringsheim's criterion:

Theorem 1.1. *The condition*

$$b_n^{-1} \text{ exists and } \left\|b_n^{-1}\right\| + \left\|a_n b_n^{-1}\right\| \leq 1, \quad n = 1, 2, \ldots, \tag{$P2$}$$

is sufficient for convergence of the continued fraction K; K then satisfies $\|K - b_0\| \leq \|I\|$. If in (P2) strict inequality holds for at least one $n \in \mathbb{N}$, we have $\|K - b_0\| < \|I\|$.

Obviously, for $\mathcal{R} = \mathbb{C}$, $(P2)$ and $(P1)$ are equivalent. In the next section, we will prove Theorem 1.1; afterwards we give an example and refer to some general areas of application. Finally, we outline further research directions.

We continue with some short remarks concerning literature about continued fractions in Banach algebras:

- The first research in this field is that due to Pfluger [19] who considered matrix continued fractions, that is $\mathcal{R} = \mathbb{C}^{m\times m}$ for some $m \in \mathbb{N}$. There is more literature dealing with continued fractions subject to definitions similar to our Definition 1.1; convergence criteria can be found in [1,7,8,20–22,25]. In some of these papers, the authors prove Pringsheim-type criteria, but in any case, they add restrictive conditions (for example $a_n \in I \cdot \mathbb{C}$ in [25]).
- The best result in the literature is due to Schelling (see [21], theorem 3) who proved that $(P2)$ and

$$\left\|a_n B_n^{-1}\right\| \cdot \|B_n\| \cdot \left\|b_n^{-1}\right\| + \left\|b_n^{-1}\right\| \leq 1, \quad n = 1, 2, \ldots, \tag{1.1}$$

  are sufficient for convergence. The sequence (B_n) is defined by $B_{-1} = 0$, $B_0 = I$ and $B_n = B_{n-1}b_n + B_{n-2}a_n$ for $n = 1, 2, \ldots$; see Lemma 2.1 below.

Schelling states that $(P2)$ guarantees the existence of B_n^{-1} in (1.1), but in his convergence proof, (1.1) is a crucial factor. In fact, for $\mathcal{R} = \mathbb{C}$, (1.1) and $(P2)$ are equivalent, but in general, (1.1) is stronger, and thus, restrictive. For example, for $\mathcal{R} = \mathbb{C}^{2\times 2}$ with row-sum norm, choose $b_n = 2I$, $a_n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Obviously $\|b_n^{-1}\| = \|a_n b_n^{-1}\| = \frac{1}{2}$, and thus, $(P2)$ holds. On the other hand, we have $B_2 = \begin{pmatrix} \frac{9}{2} & \frac{1}{2} \\ 0 & \frac{9}{2} \end{pmatrix}$, $a_2 B_2^{-1} = \frac{1}{81}\begin{pmatrix} 9 & 8 \\ 0 & 9 \end{pmatrix}$, and hence $\left\|a_2 B_2^{-1}\right\| = \frac{17}{81}$, $\|B_2\| = 5$, $\left\|b_2^{-1}\right\| = \frac{1}{2}$, yielding $\left\|a_2 B_2^{-1}\right\| \cdot \|B_2\| \cdot \left\|b_2^{-1}\right\| + \left\|b_2^{-1}\right\| > 1$, and thus, (1.1) does not hold.

- Similarly, we can consider continued fractions of the form

$$b_0 + (b_1 + (b_2 + \cdots)^{-1} c_2)^{-1} c_1.$$

In this case, $(P2)$ has to be replaced by $\|b_n^{-1}\| + \|b_n^{-1} c_n\| \leq 1$. Since this convergence criterion can be proved completely analogously, we omit further details.

- There is another generalization of continued fractions, referred to as ‘matrix continued fractions’ too; see [15]. This generalization of continued fractions is based on a slightly different idea; we omit further details here.

2. Proof of Theorem 1.1

As in the scalar case, the proof of Theorem 1.1 is based on converting the approximations $K_0^{(N)}$ into ordinary fractions $A_N B_N^{-1}$, where the numerator and denominator are subject to the same linear second-order recursion.

Lemma 2.1. *If all approximants $K_0^{(N)}$, $N = 0, 1, 2, \ldots$, exist, then $K_0^{(N)} = A_N B_N^{-1}$ holds for $N = 0, 1, 2, \ldots$, where (A_n) and (B_n) are defined by*

$$A_{-1} = I, \qquad A_0 = b_0, \qquad A_n = A_{n-1} b_n + A_{n-2} a_n \quad (n \in \mathbb{N}), \tag{2.1}$$

$$B_{-1} = 0, \qquad B_0 = I, \qquad B_n = B_{n-1} b_n + B_{n-2} a_n, \quad (n \in \mathbb{N}). \tag{2.2}$$

Since the proof has been published by various authors already (see [21] for example), it is omitted here. Lemma 2.1 directly implies a recursion for the difference between two approximations.

Lemma 2.2. *Let all approximants $K_0^{(N)}$, $N = 0, 1, 2, \ldots$, exist, and define $D_n = K_0^{(n)} - K_0^{(n-1)}$ for $n = 1, 2, \ldots$. Then*

$$D_n = -D_{n-1} B_{n-2} a_n B_n^{-1}$$

for $n = 2, 3, \ldots$, and $D_1 = a_1 b_1^{-1}$.

Proof. See [21], or use (2.1) and (2.2):

$$\begin{aligned} D_n &= -B_n^{-1}\left(A_{n-1} B_{n-1}^{-1} B_n - A_n\right) B_n^{-1} \\ &= -\left(A_{n-1} B_{n-1}^{-1} B_{n-1} b_n + A_{n-1} B_{n-1}^{-1} B_{n-2} a_n - A_{n-1} b_n - A_{n-2} a_n\right) B_n^{-1} \\ &= -\left(A_{n-1} B_{n-1}^{-1} B_{n-2} - A_{n-2}\right) a_n B_n^{-1} \\ &= -D_{n-1} B_{n-2} a_n B_n^{-1}. \end{aligned}$$

$D_1 = a_1 b_1^{-1}$ follows immediately from $A_1 = b_0 b_1 + a_1$, $B_1 = b_1$, $A_0 = b_0$ and $B_0 = I$. $\square$

As in the scalar case, we will prove that $(P2)$ implies absolute convergence of $\sum_{n=1}^{\infty} D_n$, yielding convergence of the continued fraction. In $\mathbb{C}$, the next step of the proof consists in deriving $|D_n| \leq \frac{1}{|B_{n-1}|} - \frac{1}{|B_n|}$ for $n = 1, 2, \ldots$ inductively. We will see that

$$\|D_n\| \leq \left\|B_{n-1}^{-1}\right\| - \left\|B_n^{-1}\right\| \tag{2.3}$$

holds in the general case, and that (2.3) implies absolute convergence of $\sum D_n$, but any direct induction proof for (2.3) fails. We will give a slightly stronger result below, Lemma 2.4; it is the central new idea for proving Theorem 1.1.

When proving Lemma 2.4, we make use of an easy result concerning generalizations of geometric series in Banach algebras. This familiar result is stated in Lemma 2.3; in the literature, the statement is known as a special case of a criterion for convergence of Neumann series or as a perturbation lemma, depending on the context in which it is used.

Lemma 2.3. *For* $\|s\| < 1$,

$$(I - s)^{-1} = \sum_{n=0}^{\infty} s^n$$

exists and $\left\|r(I - s)^{-1}\right\| \leq \frac{\|r\|}{1-\|s\|}$ *holds for all* $r \in \mathcal{R}$.

Proof. It is easy to check that $\sum s^n (I - s) = (I - s) \sum s^n = I$ if the series converges. For $\|s\| < 1$ and any $r \in \mathcal{R}$, the series

$$\sum_{n=0}^{\infty} \|r s^n\| \leq \|r\| \sum_{n=0}^{\infty} \|s^n\| = \frac{\|r\|}{1 - \|s\|}$$

converges in $\mathbb{R}$; the convergence of $\sum s^n$ is an easy consequence. □

Lemma 2.4. *If* $(P2)$ *holds,* B_n^{-1} *exists for all* $n \in \mathbb{N}_0$ *and*

$$\left\|B_n^{-1} B_{n-1}\right\| + \|D_n B_{n-1}\| \leq 1 \tag{2.4}$$

holds for all $n \in \mathbb{N}$. *If in* $(P2)$ $\left\|b_{n_0}^{-1}\right\| + \left\|a_{n_0} b_{n_0}^{-1}\right\| < 1$ *for some* $n_0 \in \mathbb{N}$, *strict inequality holds in* (2.4) *for* $n \geq n_0$.

Proof. For $n = 1$, we have $B_0 = I$, $B_1 = b_1$; hence B_0^{-1} and B_1^{-1} exist. According to Lemma 2.2, we have $D_1 = a_1 b_1^{-1}$, and $(P2)$ yields

$$\left\|B_1^{-1} B_0\right\| + \|D_1 B_0\| = \left\|b_1^{-1}\right\| + \left\|a_1 b_1^{-1}\right\| \leq 1.$$

Let $n \geq 2$; assume that B_{n-1}^{-1} exists and $\left\|B_{n-1}^{-1} B_{n-2}\right\| + \|D_{n-1} B_{n-2}\| \leq 1$. From (2.2) we obtain

$$B_{n-1}^{-1} B_n b_n^{-1} = I + B_{n-1}^{-1} B_{n-2} a_n b_n^{-1}.$$

Since $\left\|B_{n-1}^{-1} B_{n-2}\right\| \leq 1$ (by the induction hypothesis) and $\left\|a_n b_n^{-1}\right\| < 1$ (as follows from $(P2)$), we have $\left\|B_{n-1}^{-1} B_{n-2} a_n b_n^{-1}\right\| < 1$. Thus, Lemma 2.3 guarantees that $\left(B_{n-1}^{-1} B_n b_n^{-1}\right)^{-1}$ exists, and the inequality

$$\left\| r\left(B_{n-1}^{-1}B_nb_n^{-1}\right)^{-1}\right\| \leq \frac{\|r\|}{1-\left\|B_{n-1}^{-1}B_{n-2}a_nb_n^{-1}\right\|}$$

holds for all $r \in \mathcal{R}$. Hence, B_n^{-1} exists, and we have

$$\left\|B_n^{-1}B_{n-1}\right\| = \left\|b_n^{-1}\left(B_{n-1}^{-1}B_nb_n^{-1}\right)^{-1}\right\| \leq \frac{\left\|b_n^{-1}\right\|}{1-\left\|B_{n-1}^{-1}B_{n-2}a_nb_n^{-1}\right\|}$$

and, by means of Lemma 2.2,

$$\begin{aligned}\|D_nB_{n-1}\| &= \left\|D_{n-1}B_{n-2}a_nB_n^{-1}B_{n-1}\right\| \\ &= \left\|D_{n-1}B_{n-2}a_nb_n^{-1}\left(B_{n-1}^{-1}B_nb_n^{-1}\right)^{-1}\right\| \\ &\leq \frac{\left\|D_{n-1}B_{n-2}a_nb_n^{-1}\right\|}{1-\left\|B_{n-1}^{-1}B_{n-2}a_nb_n^{-1}\right\|}.\end{aligned}$$

By using $(P2)$ and the induction hypotheses, we finally obtain

$$\begin{aligned}\left\|B_n^{-1}B_{n-1}\right\| + \|D_nB_{n-1}\| &\leq \frac{\left\|b_n^{-1}\right\| + \|D_{n-1}B_{n-2}\| \cdot \left\|a_nb_n^{-1}\right\|}{1-\left\|B_{n-1}^{-1}B_{n-2}\right\| \cdot \left\|a_nb_n^{-1}\right\|} \\ &\leq \frac{1-\left\|a_nb_n^{-1}\right\| + \left(1-\left\|B_{n-1}^{-1}B_{n-2}\right\|\right)\cdot\left\|a_nb_n^{-1}\right\|}{1-\left\|B_{n-1}^{-1}B_{n-2}\right\| \cdot \left\|a_nb_n^{-1}\right\|} \\ &= 1.\end{aligned}$$

Obviously, $\left\|b_{n_0}^{-1}\right\| + \left\|a_{n_0}b_{n_0}^{-1}\right\| < 1$ implies $\left\|B_{n_0}^{-1}B_{n_0-1}\right\| + \left\|D_{n_0}B_{n_0-1}\right\| < 1$, and, by induction, strict inequality in (2.4) for $n \geq n_0$. □

As written above, Lemma 2.4 is the crucial idea for the proof of Theorem 1.1. The important inequality (2.3) follows immediately; for $n = 1, 2, \ldots$ we have

$$\begin{aligned}\|D_n\| &\leq \|D_nB_{n-1}\| \cdot \left\|B_{n-1}^{-1}\right\| \leq \left(1-\left\|B_n^{-1}B_{n-1}\right\|\right)\cdot\left\|B_{n-1}^{-1}\right\| \\ &\leq \left\|B_{n-1}^{-1}\right\| - \left\|B_n^{-1}\right\|.\end{aligned}$$

Of course, $\left\|b_{n_0}^{-1}\right\| + \left\|a_{n_0}b_{n_0}^{-1}\right\| < 1$ implies strict inequality here for $n \geq n_0$. Trivially, $\|D_n\| \geq 0$, and thus, $\left(\left\|B_n^{-1}\right\|\right)_{n\in\mathbb{N}_0}$ decreases monotonically. Since $\left\|B_0^{-1}\right\| = \|I\|$ and $\left\|B_n^{-1}\right\| \geq 0$, $\left\|B_n^{-1}\right\|$ converges to some $\beta \in [0, \|I\|]$. Therefore,

$$\sum_{n=1}^{\infty}\|D_n\| \leq \sum_{n=1}^{\infty}\left(\left\|B_{n-1}^{-1}\right\| - \left\|B_n^{-1}\right\|\right) = \|I\| - \beta \in [0, \|I\|],$$

and strict inequality holds if $\left\|b_{n_0}^{-1}\right\| + \left\|b_{n_0}^{-1}c_{n_0}\right\| < 1$ for some $n_0 \in \mathbb{N}$. Since $K = b_0 + \sum_{n=1}^{\infty} D_n$, this completes the proof.

3. Applications

3.1. Transformations of continued fractions

In many applications, we cannot use Theorem 1.1 directly, but have to perform a simple transformation first. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of invertible elements of $\mathcal{R}$; define $\rho_{-1} = \rho_0 = I$ and

$$\tilde{b}_n = \rho_{n-1}^{-1} b_n \rho_n, \quad n = 0, 1, 2 \ldots,$$
$$\tilde{a}_n = \rho_{n-2}^{-1} a_n \rho_n, \quad n = 1, 2, \ldots.$$

Then $K = b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}$ converges if and only if

$$\tilde{K} = \tilde{b}_0 + \tilde{a}_1 \left(\tilde{b}_1 + \tilde{a}_2 \left(\tilde{b}_2 + \cdots\right)^{-1}\right)^{-1}$$

converges. The proof is based on $\tilde{B}_n = B_n \rho_n$ and $\tilde{A}_n = A_n \rho_n$ where $\tilde{K}_0^{(N)} = \tilde{A}_N \tilde{B}_N^{-1}$; for further details we refer the reader to [21]. The transformation method can be used for deriving a convergence criterion which can be used in more applications:

Theorem 3.1. *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers. If*

$$\|I\| \le q_1 \quad \textit{and} \quad \left\|b_{n-1}^{-1}\right\| \cdot \left\|a_n b_n^{-1}\right\| \le \frac{q_n - 1}{q_{n-1} q_n}, \quad n = 2, 3, 4, \ldots \tag{P3}$$

and strict inequality holds at least for one $n \in \mathbb{N}$ (by strict inequality for $n = 1$, we understand $\|I\| < q_1$), the continued fraction

$$K = b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}$$

converges.

The proof does not differ from Schelling's derivation of theorem 4 from theorem 3 (see [21]), and therefore, we omit it.

For $K = b_0 + (b_1 + (b_2 + \cdots)^{-1} c_2)^{-1} c_1)$, we have to replace $\left\|a_n b_n^{-1}\right\|$ by $\left\|b_n^{-1} c_n\right\|$.

3.2. Example

We consider an easy application. In $\mathbb{C}$, for $|x| < 1$ we have

$$\ln(1+x) = \cfrac{x}{1 + \cfrac{x/2}{1 + \cfrac{x/6}{1 + \cfrac{2x/6}{1 + \cfrac{2x/10}{1 + \cdots}}}}} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}$$

with $b_0 = 0$, $b_n = 1$, $a_1 = x$, $a_{2k} = \frac{k}{2(2k-1)} \cdot x$, and $a_{2k+1} = \frac{k}{2(2k+1)} \cdot x$ for $n, k \ge 1$; see [18,16].

Natural logarithms can be defined for any invertible matrix; see [12]. Raissouli and Kacha [20] translated the above continued fraction expansion into the context of matrix algebras. Let $\|\cdot\|$ be some matrix norm with $\|I\| = 1$, e.g. the row-sum norm, and $A \in \mathbb{C}^{m\times m}$ with $\|A\| < 1$. Then $(I + A)^{-1}$ exists, and Raissouli and Kacha proved

$$\ln(I + A) = b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}$$

with $b_0 = 0$, $b_n = I$, $a_1 = A$, $a_{2k} = \frac{k}{2(2k-1)} \cdot A$, and $a_{2k+1} = \frac{k}{2(2k+1)} \cdot A$ for $n, k \geq 1$. For the convergence proof, they used one of their own criteria; theorem 1 in [7] and theorem 2 in [21] are appropriate too.

We can check the convergence using our Theorem 3.1: Choose $q_{2k} = 2$ for $k \geq 1$ and $q_{2k+1} = 2 - \frac{1}{k+1}$ for $k \geq 0$. Then we have $q_1 = 1$, $\frac{q_{2k}-1}{q_{2k-1}q_{2k}} = \frac{1}{2\cdot\left(2-\frac{1}{k}\right)} = \frac{k}{2(2k-1)}$ and $\frac{q_{2k+1}-1}{q_{2k}q_{2k+1}} = \frac{1-\frac{1}{k+1}}{2\cdot\left(2-\frac{1}{k+1}\right)} = \frac{k}{2(2k+1)}$ for $k \geq 1$. Thus, all inequalities in $(P3)$ hold for $\|A\| \leq 1$. For $\|A\| = 1$, we have equality for $n = 1, 2, \ldots$, and hence Theorem 3.1 does not guarantee convergence, and in fact, for some matrices A with $\|A\| = 1$, the continued fraction does not converge. For $\|A\| < 1$, strict inequality holds in $(P3)$ for $n = 2, 3, \ldots$, and Theorem 3.1 guarantees convergence. Actually, for $\|A\| = 1 - \epsilon$, we have

$$\left\|b_{n-1}^{-1}\right\| \cdot \left\|a_n b_n^{-1}\right\| \leq (1-\epsilon)\frac{q_n - 1}{q_{n-1}q_n}, \quad n = 2, 3, \ldots,$$

and this inequality is much stronger than $(P3)$.

Now, define $X(A)$ to be a solution of

$$\ln(I + A) \cdot X(A) = A. \tag{3.1}$$

If A^{-1} exists, $\ln(I + A)^{-1}$ exists too, and we have $X(A) = \ln(I + A)^{-1} \cdot A$. In the general case, there is no unique solution, but the continued fraction expansion of the logarithm,

$$\ln(I + A) = b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}) = A \cdot \left(d_0 + c_1(d_1 + c_2(d_2 + \cdots)^{-1})^{-1}\right)^{-1},$$

yields a quite natural solution:

$$X(A) = d_0 + c_1(d_1 + c_2(d_2 + \cdots)^{-1})^{-1},$$

where $d_n = I$ for $n \geq 0$, $c_{2k-1} = \frac{k}{2(2k-1)} \cdot A$, $c_{2k} = \frac{k}{2(2k+1)} \cdot A$ for $k \geq 1$. We check the convergence of this new continued fraction expansion using Theorem 3.1. Choose $q_{2k-1} = 2$ and $q_{2k} = 2 - \frac{1}{k+1}$ for $k \geq 1$. Then we have $q_1 = 2$, $\frac{q_{2k}-1}{q_{2k-1}q_{2k}} = \frac{1-\frac{1}{k+1}}{2\cdot\left(2-\frac{1}{k+1}\right)} = \frac{k}{2(2k+1)}$ for $k \geq 1$ and $\frac{q_{2k-1}-1}{q_{2k-2}q_{2k-1}} = \frac{1}{2\cdot\left(2-\frac{1}{k}\right)} = \frac{k}{2(2k-1)}$ for $k \geq 2$. Thus, for $\|A\| \leq 1$, $(P3)$ holds with $\|I\| = 1 < q_1$, and therefore, the continued fraction $d_0 + c_1(d_1 + c_2(d_2 + \cdots)^{-1})^{-1}$ converges.

For $\|A\| = 1$, the convergence is not guaranteed by the convergence criteria in [20,7,21], whereas it is an easy application of Theorem 3.1. The case $\|A\| = 1$ is interesting for two reasons:

- Even for $\|A\| = 1$, $(I + A)^{-1}$ may exist, and then $\ln(I + A)$ exists too. In this case, (3.1) has a unique solution; it is given by the continued fraction expansion above.
- The function $F : A \mapsto \ln(I + A)^{-1} \cdot A$ is defined on

  $$E = \{A : \|A\| < 1,\ A^{-1} \text{ exists}\}.$$

  $X : A \mapsto X(A) = d_0 + c_1(d_1 + c_2(d_2 + \cdots)^{-1})^{-1}$ gives a continuation defined on the **closed** unit circle $\{A : \|A\| \leq 1\}$ (in which E is dense).

3.3. More applications

Continued fractions in Banach algebras, especially matrix continued fractions, are used in many areas of application. An important reason is that solutions of difference equations can be characterized by means of continued fractions. In $\mathbb{C}$, for many choices of coefficients, the system of difference equations

$$x_n = b_n x_{n+1} + a_{n+1} x_{n+2}, \quad n = 0, 1, 2, \ldots$$

is solved by two non-trivial sequences (x_n) and (y_n) satisfying $\lim \frac{x_n}{y_n} = 0$. In this case, (x_n) is called the sub-dominant solution. It turns out (see [18], chapter II, Section 20) that, under some mild constraints, the sub-dominant solution can be determined in terms of continued fractions, that is

$$\frac{x_0}{x_1} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}},$$

Gautschi's algorithm [9] is based on this fact.

Similarly, solutions of second-order vector–matrix difference equations can be characterized by using continued fractions. An application can be found in [2]; the author uses a convergence criterion from [7]. Continued fractions are implicitly used when solving for the stationary distribution (characterized by vector–matrix difference equations) of Markov chains with a special structure, so-called quasi-birth–death processes, by means of matrix-analytic methods (see [3]). The relationship to continued fraction is explicitly pointed out in [10].

Naturally, there are more applications of continued fractions and their generalizations. For example, there is a similar relationship between second-order linear differential equations and continued fractions (see [13], chapter 7, for an introduction), and matrix continued fractions are applied to matrix-valued Riccati differential equations; see [5].

In complex analysis, there are some interesting results concerning applications of continued fractions (see [24] for an introduction), and when solving equations in Banach algebras, in particular quadratic equations, sometimes continued fractions are appropriate; see [4].

4. Conclusions and further research

The central novelty of this paper is the proof that the Pringsheim-type condition $(P2)$ is sufficient for convergence of the continued fraction

$$b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}.$$

In comparison to the proof in the scalar case, the proving technique has not changed in general; we still make use of the recursion for both numerator and denominator, but some details have to be modified.

There are some further research directions. In $\mathbb{C}$, de Bruin [6] introduced generalized continued fractions, GCFs; they can be used for solving difference equations of higher order (see [23,11]) and extending Miller's algorithm [17]. For GCFs in $\mathbb{C}$, there is an appropriate generalization of Pringsheim's criterion; see [14]. A natural task is defining GCFs in Banach algebras and extending Pringsheim's criterion to GCFs in Banach algebras.

Another question concerns 'two-sided' continued fractions in Banach algebras, that is

$$b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1} c_2)^{-1} c_1.$$

Due to the absence of commutativity, in general, there is no chance of eliminating either all a_n or all c_n. For example, when applying continued fraction theory to difference equations one often has to deal with two-sided continued fractions; see [2] for example.

In the literature, there are some Pringsheim-type criteria for two-sided continued fractions (see [7,21]), but the authors add quite restrictive conditions. The problem when considering two-sided continued fractions is that there is no simple conversion into ordinary fractions $A_N B_N^{-1}$, $B_N^{-1} C_N$ or $A_N B_N^{-1} C_N$ for the Nth approximation, so an exact generalization of Pringsheim-type convergence criteria requires a new proving technique.

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